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Multiple scattering in the presence of absorption: a theoretical treatment for quasi one-dimensional systems

N A Bruce and J T Chalker

Theoretical Physics, University of Oxford,1 Keble Rd, Oxford OX1 3NP, UK

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Abstract. We consider propagation of waves in a disordered medium, extending the treatment of quasi one-dimensional systems, due to Dorokhov and to Mello *et al* to include absorption. In particular, we obtain within this approach the probability distribution of the reflection matrix for a semi-infinite system, as a function of the ratio of the mean free paths for absorption and scattering.

1. Introduction

There has been sustained theoretical and experimental interest [1] during the last decade in the multiple scattering of classical waves in both ordered and random media. From an early stage [2], extensive use has been made of analogies between phenomena occurring for classical waves and their counterparts in disordered electronic systems [3]. These phenomena include the existence of photonic band gaps, speckle patterns and enhanced backscattering [1].

In studies of scattering in random media, an important objective is to understand the statistical properties of reflected and transmitted waves, as a function of system geometry and disorder strength. Fluctuations in these quantities are responsible for speckle patterns in the optical context, and for conductance fluctuations in mesoscopic conductors [1]. Scattering of electromagnetic waves is of particular interest in connection with fluctuations, because it is possible experimentally [4] to get more detailed information for this case than for electron transport. Specifically, spatial correlations in intensity can be obtained, whilst from conductance measurements one learns only about integrated transmission probabilities.

Many different theoretical approaches have been applied to these problems, including diagrammatic methods [5], Langevin techniques [6] and random matrix theory [7–10]. The last of these, although it involves strong simplifying assumptions, provides a framework within which a reasonably complete description of statistical properties can be derived. The version of random matrix theory with the clearest microscopic foundations is due to Dorokhov [8] and to Mello *et al* [9]. It has at its centre a Fokker–Planck equation (known as the DMPK equation) for the evolution of scattering properties with sample length. The main aim of the present paper is to generalize this approach to include absorption as well as multiple scattering.

The combined consequences of absorption and multiple scattering for fluctuations in the transmission properties of disordered waveguides have been studied previously using other methods, by Stephen [11], and by Pnini and Shapiro [12], who were able to account in detail [13] for microwave measurements. The transmission behaviour that has been the

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focus of this earlier work turns out to be difficult to obtain from a Fokker–Planck equation. Instead we concentrate on reflection properties. We consider systems in which the number of scattering modes, N, is arbitrary. The special case of of a strictly one-dimensional system $N = 1$ has been solved recently by Freilikher *et al* [14]

In the following, we define our model in section 2, derive a Fokker–Planck equation in section 3, and discuss its solution in section 4.

2. A model for multiple scattering in the presence of absorption, in quasi one-dimensional systems

In this section we generalize the quasi one-dimensional scattering model of Dorokhov [8] and Mello *et al* [9] to include absorption. The model provides an idealized description of a waveguide or optical fibre along which N modes can propagate in each direction. Our aim is to calculate the scattering properties of such a system as a function of its length. To this end, first consider the scattering and absorption that occur in a section of infinitesimal length δl. Wave amplitudes in each mode and on each side of this section are related by an S-matrix. Let the N amplitudes of waves incident in each mode from one side form the components of a vector a_l. Similarly, let the outgoing wave amplitudes on the other side be the components of a_r; and let b_l and b_r, respectively, be the outgoing and incident amplitudes for waves travelling in the opposite direction. In terms of the $2N \times 2N$ S-matrix

$$\begin{pmatrix} b_l \\ a_r \end{pmatrix} = S \begin{pmatrix} a_l \\ b_r \end{pmatrix}. \tag{1}$$

Since δl is infinitesimal, we expect that $a_r \simeq a_l$ and $b_l \simeq b_r$. It is therefore convenient to define a transformed S-matrix, S', which is close to the $2N \times 2N$ unit matrix. Let

$$S' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S \equiv \begin{pmatrix} \tau & \rho' \\ \rho & \tau' \end{pmatrix} \tag{2}$$

where ρ and τ are the reflection and transmission matrices for waves incident from (say) the left, and ρ', τ' are the corresponding matrices for waves incident from the right. S' can be parametrized as $S' = \exp(\mathrm{i}X)$, with X infinitesimal. Without absorption, S, and hence S', is unitary, and so X is Hermitian. In the presence of absorption it is thus natural to separate X into Hermitian and anti-Hermitian parts by writing $X = H + \mathrm{i}G$, where H and G are $2N \times 2N$ Hermitian matrices. We treat scattering and absorption as independent processes, and so discuss the properties of H and G separately. Supposing that scattering processes in isolation are time-reversal symmetric, H has the form [15]

$$H = \mu \begin{pmatrix} x & y \\ y^* & x^* \end{pmatrix} \tag{3}$$

where x is an $N \times N$ Hermitian matrix and y is an $N \times N$ symmetric matrix. The infinitesimal μ is related to the length δl of the section of waveguide under consideration: since the scattering *probability*, which is of order $|\mu|^2$, should be proportional to length, we require the scattering *amplitude* to be of order $\mu \equiv (\delta l)^{1/2}$.

To model random scattering, we take x and y statistically independent in successive sections. For mathematical simplicity, we follow Dorokhov [8] and Mello *et al* [9], and consider quasi one-dimensional systems in which the distributions of x and y are invariant under arbitrary unitary transformations that mix the N modes of the waveguide. Physically, this means that we ignore any structure in the directions transverse to the length of the waveguide: it should be a good approximation provided the transverse dimensions are not greater than the mean free path for scattering. Because we take δl infinitesimal,

it is necessary only to specify the first and second moments of x and y. We set [15] $\langle x\rangle = \langle y\rangle = \langle xy\rangle = \langle yy\rangle = 0$ and

$$\langle y_{\alpha\beta} y^*_{\gamma\delta}\rangle = \frac{\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}}{N+1} \tag{4}$$

$$\langle x_{\alpha\beta} x^*_{\gamma\delta}\rangle = \frac{\delta_{\alpha\gamma}\delta_{\beta\delta}}{N} \tag{5}$$

where the subscripts are channel indices. Note that, with this choice, the mean free path for scattering is implicitly taken as the unit of length.

Turning to absorption, if the outgoing flux is never to be greater than the incoming flux, it is necessary that the eigenvalues of $S^\dagger S \equiv S'^\dagger S'$ have modulus $\leqslant 1$. At lowest order in δl, this implies that the eigenvalues of $\mathrm{i}(X - X^\dagger) \equiv -2G$ should be negative. Since the probability for absorption should be proportional to the length $\delta l \equiv \mu^2$, and since we wish to retain the invariance of the model under unitary transformations that mix modes, we set

$$\langle G\rangle = a\mu^2 \mathbf{1} \tag{6}$$

where $\mathbf{1}$ denotes the $2N \times 2N$ unit matrix and a parametrizes the strength of absorption: specifically, $1/a$ is the absorption length in units of the scattering mean free path. The constraint that the eigenvalues of $-2G$ are negative for every realization of G, together with the fact that the first moment is of order μ^2, implies that higher cumulants of G are of higher order in μ, and therefore irrelevant in the limit $\mu \to 0$.

Summarizing, to order μ^2 we obtain from our definition of S' and expressions for H and G

$$S' = \begin{pmatrix} \mathbf{1} + \mathrm{i}\mu x - \mu^2 a\mathbf{1} - \frac{1}{2}\mu^2(yy^* + x^2) & \mathrm{i}\mu y - \frac{1}{2}\mu^2(xy + yx^*) \\ \mathrm{i}\mu y^* - \frac{1}{2}\mu^2(y^*x + x^*y^*) & \mathbf{1} + \mathrm{i}\mu x^* - \mu^2 a\mathbf{1} - \frac{1}{2}\mu^2(y^*y + x^{*2}) \end{pmatrix} \tag{7}$$

in which $\mathbf{1}$ is the $N \times N$ unit matrix.

3. Fokker–Planck equation for the evolution of the reflection matrix with system length

We next derive a Fokker–Planck equation for the evolution with system length of the reflection matrix of the model introduced above. One can, of course, describe scattering for the system as a whole using a $2N \times 2N$ S'-matrix similar to that for a short section. It consists of four $N \times N$ blocks:

$$S' = \begin{pmatrix} t & r' \\ r & t' \end{pmatrix} \tag{8}$$

where r, t, r' and t' are reflection and transmission matrices for waves incident from each side, in analogy with (2). In a system with time-reversal symmetry and no absorption, the reflection and transmission matrices can be parametrized as [9]

$$\begin{aligned} r &= -v^{\mathrm{T}}\Lambda^{1/2}v \qquad & t &= u\Delta^{1/2}v \\ r' &= u\Lambda^{1/2}u^{\mathrm{T}} \qquad & t' &= v^{\mathrm{T}}\Delta^{1/2}u^{\mathrm{T}} \end{aligned} \tag{9}$$

where u and v are unitary, and Λ and Δ are diagonal, with $\Lambda + \Delta = 1$ from flux conservation.

In the presence of absorption, the situation is more complicated. In particular, the unitary matrices appearing in the polar decomposition of t and t' are, in general, different from those appearing in the decomposition of r and r'. As a result we have been unable to calculate the transmission properties of our model. The reflection properties, however,

can be obtained in the following way, from an extension of the Dorokhov–Mello–Peyrera–Kumar [8, 9] (DMPK) equation. Consider r': we shall show that the decomposition of (9) remains appropriate in a system with absorption and derive a Fokker–Planck equation for the evolution with system length of the diagonal elements, Λ_α, of Λ.

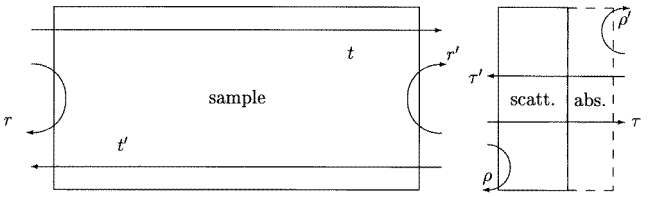

Figure 1. Picture of the model showing the addition of an extra slice to a sample.

Imagine increasing the system length by an amount δl, adding an extra section from the right, as sketched in figure 1. The reflection matrix, r', of the longer system is given by the multiple scattering series

$$r'_f = r' + \delta r' = \rho' + \tau(r' + r'\rho r' + \cdots)\tau'. \tag{10}$$

Substituting for ρ', τ and τ' from (2), we obtain

$$\begin{aligned} r'_f = r' + \delta r' = r' + \mathrm{i}\mu[y + xr' + r'x^* + r'y^*r'] \\ +\mu^2[-2ar' - \tfrac{1}{2}(yy^* + x^2)r' - \tfrac{1}{2}r'(y^*y + x^{*2}) - xr'x^*] \end{aligned} \tag{11}$$

where we have retained all terms to order μ, and those at order μ^2 that have a non-zero average; omitted terms make no contribution to the Fokker–Planck equation we derive. Note that, if $r'^{\mathrm{T}} = r'$, then $r'^{\mathrm{T}}_f = r'_f$, so that if $r' = u\Lambda^{1/2}u^{\mathrm{T}}$, then r'_f can be written similarly as $r'_f = u_f\Lambda_f^{1/2}u_f^{\mathrm{T}}$. Let $\delta\Lambda = \Lambda_f - \Lambda$. We use a perturbative calculation to order μ^2 to calculate $\delta\Lambda$ from the eigenvalues of $r'^{\dagger}_f r'_f$ and average the result, which is a function of x and y, according to (4), to obtain

$$\langle\delta\Lambda_\alpha\rangle = \mu^2\left[-4a\Lambda_\alpha + \frac{2}{N+1}(1-\Lambda_\alpha)^2 + \frac{2}{N+1}\sum_{\beta\neq\alpha}\frac{\Lambda_\alpha(1-\Lambda_\alpha)(1-\Lambda_\beta)}{\Lambda_\alpha - \Lambda_\beta}\right] \tag{12}$$

and

$$\langle\delta\Lambda_\alpha\delta\Lambda_\beta\rangle = \mu^2\frac{4}{N+1}\delta_{\alpha\beta}\Lambda_\alpha(1-\Lambda_\alpha)^2. \tag{13}$$

By standard arguments (see, for example, [16]) the probability distribution of $\{\Lambda_\alpha\}$ evolves with sample length, L, according to a Fokker–Planck equation with drift and diffusion coefficients, $\tilde{D}_\alpha$ and $\tilde{D}_{\alpha,\beta}$ given by

$$\tilde{D}_\alpha = \frac{1}{\mu^2}\langle\delta\Lambda_\alpha\rangle \qquad \tilde{D}_{\alpha\beta} = \frac{1}{2\mu^2}\langle\delta\Lambda_\alpha\delta\Lambda_\beta\rangle. \tag{14}$$

For ease of comparison with previous work, we transform to the variables $\{\lambda_\alpha\}$, defined by $\Lambda = \lambda/(1+\lambda)$, and introduce a rescaled system length, $t = L/(N+1)$. In these variables, the drift and diffusion coefficients, $\hat{D}_\alpha$ and $\hat{D}_{\alpha\beta}$ are

$$\hat{D}_\alpha = -4a(N+1)\lambda_\alpha(1+\lambda_\alpha) + 2(1+2\lambda_\alpha) + 2\sum_{\beta\neq\alpha}\frac{\lambda_\alpha(1+\lambda_\alpha)}{\lambda_\alpha - \lambda_\beta} \tag{15}$$

$$\hat{D}_{\alpha\beta} = 2\delta_{\alpha\beta}\lambda_\alpha(1+\lambda_\alpha). \tag{16}$$

The joint probability distribution of $\{\lambda_\alpha\}$, $W(\lambda)$, hence evolves with system length t according to

$$\frac{\partial W}{\partial t} = \left[-\sum_\alpha \frac{\partial}{\partial \lambda_\alpha} \hat{D}_\alpha + \sum_{\alpha\beta} \frac{\partial^2}{\partial \lambda_\alpha \partial \lambda_\beta} \hat{D}_{\alpha\beta} \right] W. \tag{17}$$

Without absorption ($a = 0$), this reduces to the DMPK equation of [8, 9].

4. Stationary solution of the Fokker–Planck equation

The Fokker–Plank equation, (17), has a stationary solution in the limit of long samples ($t \to \infty$). In this limit, the matrix u appearing in the polar decomposition of r' (equation (9)) has a uniform distribution on $U(N)$. This fact combined with knowledge of the distribution, $W(\lambda)$, provides a complete characterization of scattering properties. This limit is, of course, non-trivial only in the presence of absorption: without it, all λ_α diverge with t, $\Lambda \to 1$ and $r' \to uu^{\mathrm{T}}$.

The stationary solution is obtained most transparently by transforming to variables in which the diffusion coefficient is constant, rather than a function of the coordinates. We therefore substitute for $\{\lambda_\alpha\}$ in terms of $\{x_\alpha\}$ via

$$\lambda_\alpha = \tfrac{1}{2}[\cosh(2x_\alpha) - 1]\,. \tag{18}$$

In these coordinates the drift and diffusion coefficients, D_α and $D_{\alpha\beta}$ are

$$D_\alpha = -\frac{1}{2}\frac{\partial V}{\partial x_\alpha} \tag{19}$$

and

$$D_{\alpha\beta} = \tfrac{1}{2}\delta_{\alpha\beta} \tag{20}$$

where

$$V = \sum_\alpha U_1(x_\alpha) + \sum_{\alpha<\beta} U_2(x_\alpha, x_\beta) \tag{21}$$

with

$$U_1(x) = [a(N+1)\cosh(2x) - \ln|\sinh(2x)|] \tag{22}$$

and

$$U_2(x, y) = \ln|\cosh(2x) - \cosh(2y)|. \tag{23}$$

It follows immediately [16] that the limiting distribution, $W'(x)$ is

$$W' = \exp(-V). \tag{24}$$

Distributions of this kind have been studied extensively in connection with random matrix theory and, in particular, the global maximum entropy approach to disordered conductors [7]. In the terminology of that approach, equations (21) and (24) describe the statistical mechanics of N classical particles moving in one dimension, with coordinates $\{x_\alpha\}$, under the influence of a one-body confining potential, $U_1(x)$, and a repulsive two-body interaction, $U_2(x, y)$. Since the interaction has the same form as occurs in the global maximum entropy approach, our result corresponds in that language simply to a particular choice of one-body potential. It should be emphasized, however, that whilst the global maximum entropy approach provides only an approximate treatment of scattering without absorption, equation (24) is an *exact* solution for our model of scattering in the presence of absorption.

As $W'(x)$ is a joint distribution for N variables, some further effort is required to obtain from it physically observable quantities. To do so, we take over an approach developed in random matrix theory. Suppose that absorption is not too weak and the number of channels is large, so that $aN \gg 1$. Then one expects the x_α's typically to be closely spaced, with a density $N\rho(x)$. In this continuum approximation

$$V = N\int_0^\infty U_1(x)\rho(x)\,\mathrm{d}x + \frac{N^2}{2}\int_0^\infty\int_0^\infty \rho(x)U_2(x,y)\rho(y)\,\mathrm{d}x\,\mathrm{d}y. \tag{25}$$

The most probable density, ρ_0, is the one that minimizes V

$$\left.\frac{\partial V}{\partial \rho}\right|_{\rho=\rho_0} = 0. \tag{26}$$

From this we obtain an integral equation for $\rho_0(x)$, satisfied in the range, $0 < x < x_{\max}$, over which $\rho_0(x)$ is non-zero. Neglecting terms in $U_1(x)$ small compared to aN

$$\int_0^\infty \mathrm{d}y\,\frac{\rho_0(y)}{\cosh(2x)-\cosh(2y)} = a. \tag{27}$$

The solution of this equation can be written most simply after transforming back to the coordinates λ; in these variables we find $\rho_0(\lambda) = 0$ for $\lambda < 0$ or $\lambda > \lambda_{\max}$ and

$$\rho_0(\lambda) = \frac{2a}{\pi}\left(\frac{a^{-1}-\lambda}{\lambda}\right)^{1/2} \tag{28}$$

for $0 < \lambda < \lambda_{\max}$, with $\lambda_{\max} = 1/a$. We note that the expected behaviour in the absence of absorption, $\lambda_\alpha \to \infty$, is recovered for $a \to 0$: $\lambda_{\max}$ diverges and $\rho_0(\lambda)$ falls to zero. Conversely, in the strong-absorption limit, $a \to \infty$, all λ_α are zero, and the reflection matrix vanishes. The continuum treatment should be reliable provided $N\rho_0(\lambda) \gg 1$ in the region where it is non-zero: the condition for this is $aN \gg 1$, as anticipated.

Various physical quantities can be calculated from the limiting density, $\rho_0(\lambda)$. For example, the reflection probability (into any outgoing channel) for a wave incident (in channel α) on a semi-infinite sample is

$$R = \sum_\beta |r'_{\alpha\beta}|^2. \tag{29}$$

Its average value is (within the continuum approximation)

$$\langle R\rangle = \int_0^\infty \rho_0(\lambda)\frac{\lambda}{1+\lambda}\,\mathrm{d}\lambda \tag{30}$$

from which we obtain

$$\langle R\rangle = 1 + 2a - 2(a+a^2)^{1/2}. \tag{31}$$

The reflection probability falls to zero for strong absorption, as $\langle R\rangle \sim 1/4a$, since in this limit any reflection that takes place must occur within a surface layer with thickness of the order of an absorption length, $1/a$, and the probability for this to happen is proportional to the thickness. With weak absorption, the reflection probability approaches 1, as $1-\langle R\rangle \sim 2a^{1/2}$. To interpret this result [2], consider first a system of *finite* length, *without* absorption. Recall that the reflection probability, $R(L)$, for a non-absorbing system of length L satisfies $1 - R(L) = l/L$, where l is the mean free path for scattering, provided localization effects are unimportant. A reflected wavepacket that propagates diffusively within the system will travel a maximum distance of order L^2/l before reflection. Suppose now that absorption occurs, with an absorption length $1/a$. R will be almost unchanged if $L^2/l \ll 1/a$; on increasing L, the crossover from reflection limited by sample length to reflection limited

by absorption occurs at $L^2/l \sim 1/a$. At this length, $1 - R \sim [al]^{1/2}$, which is the result obtained from (31) since, in defining the model, we have used l as the unit of length.

Sample-to-sample fluctuations in reflection properties are also calculable using established methods. Within the continuum treatment, one can show with the approach of Beenakker [17] that fluctuations of the density $\rho(\lambda)$ about the most probable value, $\rho_0(\lambda)$, are Gaussian. The simplest physical quantity influenced by these fluctuations alone is the reflection probability, summed over both ingoing and outgoing channels

$$\hat{R} = \sum_{\alpha\beta} |r'_{\alpha\beta}|^2. \tag{32}$$

Its fluctuations are closely analogous to fluctuations of the conductance in mesoscopic conductors. For $N \gg 1$ and $a \ll 1$, the results of [17] imply a value for the variance of $\hat{R}$ that is independent of $\langle R \rangle$

$$\langle [\hat{R} - \langle \hat{R} \rangle]^2 \rangle = \tfrac{1}{8}. \tag{33}$$

5. Summary

In this paper we have shown how the DMPK equation is modified in the presence of absorption. Absorption changes profoundly the behaviour of the solution to the DMPK equation. Without absorption, no limiting solution exists and the variables parametrizing scattering diverge with system length. With absorption, the equation has a non-trivial limiting solution. For a system with N scattering channels, this solution is a function of N variables and characterizes completely the reflection properties in the semi-infinite limit. Remarkably, it is possible to obtain the limiting solution explicitly. From it, we have calculated, for systems with many channels and weak absorption, the mean and variance of the reflection probability. We leave as an interesting open problem the calculation of transmission amplitudes for the same model, which, with absorption, are no longer simply related to the reflection properties.

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